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Short Communication

Influence of hysteretic dissipation on chaotic responses

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1. Introduction

Hysteresis, a phenomenon well-known in many fields of science, is caused by various processes. The models describing systems with hysteresis are discontinuous and contain high nonlinearities with memory-dependent properties. Investigation of these systems within the framework of approximate analytical approaches, such as slowly varying parameters or harmonic balance methods, admits the conclusion that independently of the values of control parameters a hysteretic system under external periodic excitation exhibits a stable symmetric asymptotic response. However, recent publications and research results, based on both numerical and combined numerical–analytical techniques, show frequency–response curves and bifurcation diagrams that point to the presence of solutions and bifurcations mostly unexpected for hysteretic oscillators. At the same time, studies into control parameter spaces are evidently insufficient. In this connection, prediction of the behaviour of such systems depending on various parameters becomes highly topical.

In hysteresis simulation a system is frequently viewed as a black box and the system's output (or response) is modelled with the use of analytical expressions or differential equations supposing that the input of the system is known. Though various transient processes of the input may be reflected in the formation of minor loops, as a rule it is the regular signals of inputs that are considered and the regular response of the system is usually expected. However, as the

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investigation shows, the hysteretic dissipation can change the conditions for the chaotic response occurrence in a substantial way.

In the present work the classical Masing and Bouc–Wen hysteretic oscillators are considered. The motion of each oscillator is described by a coupled differential system and the motion components exert influence on each other in time. The restraining and generating effects of the hysteretic dissipation on the chaotic behaviour occurrence are demonstrated. To predict conditions for the stable/unstable behaviour of the hysteretic oscillators under harmonic excitation an effective algorithm aiming at the numerical analysis of these systems is applied. This technique is based on the analysis of the wandering trajectories and has already been successfully applied to the cases of smooth and non-smooth systems (cf. Refs. [1–4]). From the computational point of view, the applied approach is simpler and faster than the standard procedures (the theory, implementation and sample numerical algorithms are presented, for example, in Refs. [5–9]). The problems and difficulties of principle which arise using standard procedures are also reported in these references.

According to Wolf's algorithm, the calculation of the Lyapunov exponent λ as the measure of the trajectory divergence begins with the choosing of a basic trajectory $\mathbf{x}^*(t, \mathbf{x}^{(0)})$. At each time step t_k the dynamical system $\dot{\mathbf{x}} = f(t, \mathbf{x})$ under investigation, where $\mathbf{x} \in R^n$ is the state vector, $f(t, \mathbf{x})$ is defined in $R \times R^n$ and is describing the time derivative of the state vector, is integrated again with any neighbouring points $\mathbf{x}^*(t_k) + \eta$ acting as the initial conditions. Thus, to find λ the governing equations and the corresponding variational equations $\dot{\eta} = \mathbf{A} \cdot \eta$, in which $\mathbf{A}$ is matrix of partial derivatives $\nabla f(\mathbf{x}^*(t_k))$, are solved N times (where N is the number of the time steps). Averaging over a long time results in a reliable value of λ variations of distances between the trajectories. The analogous calculations it is necessary to execute for all nodal points of a sampled space under investigation. This procedure is very computationally intensive especially for discontinuous systems. The method can often suffice for low-dimensional systems, but in practice, it fails with a chaotic system of high-dimensions. To realize this paper's approach it is enough to solve the equations governing the dynamical system only two times for each selected trajectory. The governing equations are solved 2 times instead of $2N$ times and it is not required to average the variations of distances between the trajectories over a long time.

2. Results and discussions

Let us consider classical hysteretic models, like the Masing oscillator [10–12] and the Bouc–Wen oscillator [12–14]. In both cases, external periodic excitation with amplitude F and frequency Ω acts on mass m which oscillates along an inertial base. These oscillators possess hysteretic properties and it is assumed that there is a linear viscous damper with coefficient 2μ.

The following set of differential equations governs the motion of the Masing oscillator:

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -2\mu y - (1 - v)g(x) - vz + F \cos \Omega t, \\
\dot{z} &= g'\left(\frac{z - z_i}{2}\right)y.
\end{aligned}
\tag{1}
$$

In the above $v \in [0,1]$, $g(x) = (1-\delta)x/(1+|x|^n)^{1/n} + \delta x$, $R = (1-v)g(x) + vz$ is the total restoring force with nonlinear elastic part $(1-v)g(x)$ and hysteretic part vz. It is clear that the case $v = 1$ corresponds to the maximum hysteretic dissipation and $v = 0$ corresponds to the elastic behaviour of the oscillator. Parameter δ characterizes the ratio between the post- and pre-yielding stiffness. Parameter n governs smoothness of the transitions from the elastic to the plastic range. Couples $\pm(x_i, z_i)$ represent the velocity reversal points at $\dot{x} = 0$. According to the Masing rule extended onto the case of the steady-state motion of the hysteretic oscillator, loading/unloading branches of a hysteresis loop are geometrically similar. Thus, if $f(x,z) = 0$ is the equation of a virgin loading curve, then the equations $f((x \pm x_i)/2, (z \pm z_i)/2) = 0$ describe loading/unloading branches of the hysteresis loop. In the case of the non-steady state motion of the Masing oscillator it is assumed that the equation of any hysteretic response curve can be obtained by applying the original Masing rule to the virgin loading curve using the latest point of the velocity reversal.

The motion of the Bouc–Wen oscillator is governed by the following set of differential equations:

$$\dot{x} = y,$$

$$\dot{y} = -2\mu y - \delta x - (1-\delta)z + F\cos\Omega t,$$

$$\dot{z} = [k_z - (\gamma + \beta\,\mathrm{sgn}(\dot{x})\mathrm{sgn}(z))|z|^n]y, \tag{2}$$

where $R = \delta x + (1-\delta)z$ is the total restoring force; parameters $(k_z, \beta, n) \in R^+$ and $\gamma \in R$ govern the shape of the hysteresis loop. Parameters δ and n have the same sense as in the case of the Masing model.

The chaotic behaviour of nonlinear deterministic systems assumes the wandering of the motion trajectories around various equilibrium states. They are characterized by unpredictability and sensitive dependence on the initial conditions. By analyzing the motion trajectories of these systems, it is possible to find the chaotic vibration regions in the control parameter space.

For the sake of chaotic and regular dynamics tracing, it is supposed that with the increase of time all trajectories remain in the closed bounded domain of the phase space. To analyze trajectories of sets (1) and (2), characteristic vibration amplitudes A_i of the motion components are introduced $A_i = \frac{1}{2}|\max_{t_1 \leqslant t \leqslant T} x_i(t) - \min_{t_1 \leqslant t \leqslant T} x_i(t)|$. Here and below index number, i runs over three values corresponding to three generalized coordinates x, y, z. $[t_1, T] \subset [t_0, T]$ and $[t_0, T]$ is the time interval, in which the trajectory is considered, and $[t_0, t_1]$ is the time interval in which all transient processes are damped. Two neighbouring initial points $\mathbf{x}^{(0)} = \mathbf{x}(t_0)$ and $\tilde{\mathbf{x}}^{(0)} = \tilde{\mathbf{x}}(t_0)$ ($\mathbf{x} = (x,y,z)^{\mathrm{T}}$ or $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$) are chosen in the three-dimensional parallelepiped $P_{\delta_x,\delta_y,\delta_z}(\mathbf{x}^{(0)})$ such that $|x_i^{(0)} - \tilde{x}_i^{(0)}| < \delta_i$, where $\delta_i > 0$ is small in comparison with A_i. In the case of regular motion it is expected that the $\varepsilon_i > 0$ used in inequality$|x_i(t) - \tilde{x}_i(t)| < \varepsilon_i$ is also small in comparison with A_i. The wandering orbits attempt to fill up some bounded domain of the phase space. At instant t_0 the neighbouring trajectories diverge exponentially afterwards. Hence, for some instant t_1 the absolute values of differences $|x_i(t) - \tilde{x}_i(t)|$ can take any values in closed interval $[0, 2A_i]$. An auxiliary parameter α is introduced, $0 < \alpha < 1$. αA_i is referred to as divergence measures of observable trajectories in the directions of generalized coordinates, and with the aid of parameter α one has been chosen, which is *inadmissible* for the case of 'regularity' of the motion. The domains, where a chaotic behaviour of the considered systems is possible, can be found using the following condition:

$$\exists t^* \in [t_1, T] : |x(t^*) - \tilde{x}(t^*)| > \alpha A_x. \tag{3}$$

If inequality (3) is satisfied in some nodal point of the sampled control parameter space, then such motion is relative to chaotic one (including transient and alternating chaos). The manifold of all such nodal points of the investigated control parameter space set up chaotic behaviour domains for the considered systems.

The motion stability depends on all parameters of the considered hysteretic models including initial conditions. Irregular responses of the Masing and the Bouc–Wen hysteretic oscillators in the planes defined by the damping coefficient—amplitude and the frequency—amplitude of the external periodic excitation have been traced with a sufficient accuracy after a coordinate sampling. Figs. 1 and 2 display the relevant domains.

In the case of the elastic behaviour without hysteretic dissipation ($v = 0$) the Masing oscillator (1) is nonlinear and its chaotic behaviour has been detected. It is worth noticing, that for the viscous damping equal to zero ($\mu = 0$) (Fig. 1(a)), the domains of the chaotic behaviour in the (Ω, F) plane decrease with the increase of the hysteretic dissipation value. In this case the hysteretic dissipation has a *restraining effect*. When $\mu \neq 0$ (Fig. 1(b)) the hysteretic dissipation changes the form and location of the chaotic domains in the (μ, F) plane that made conditional upon mutual influence of the nonlinear terms in set (1).

The situation is different for the Bouc–Wen oscillator which is linear (when $\delta = 0$). Therefore, chaotic responses of the Bouc–Wen oscillator are not observed up to some value of hysteretic dissipation δ_{cr}, when the influence of the nonlinear terms becomes critical. In other words, adding hysteretic dissipation leads to chaotic responses occurring in this system (Fig. 2(a) and (b)). It demonstrates *generating effect* of the hysteretic dissipation on chaos occurring in hysteretic system.

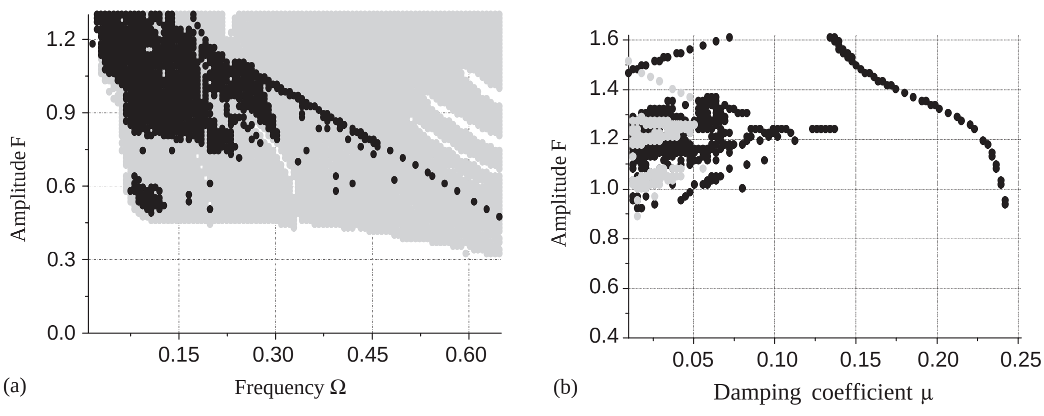

Fig. 1. Domains where chaotic behaviour of the Masing histeretic oscillator is possible ($\delta = 0.05$, $n = 10.0$, $x(0) = 0.1$, $\dot{x}(0) = 0.1$, $z(0) = 0$): (a) in the (Ω, F) plane ($\mu = 0$); (b) in the (μ, F) plane ($\Omega = 0.15$). Grey colour corresponds to the pure elastic behaviour of the oscillator without any hysteretic dissipation ($v = 0$). Black colour corresponds to the motion with the hysteretic dissipation value $v = 0.5$.

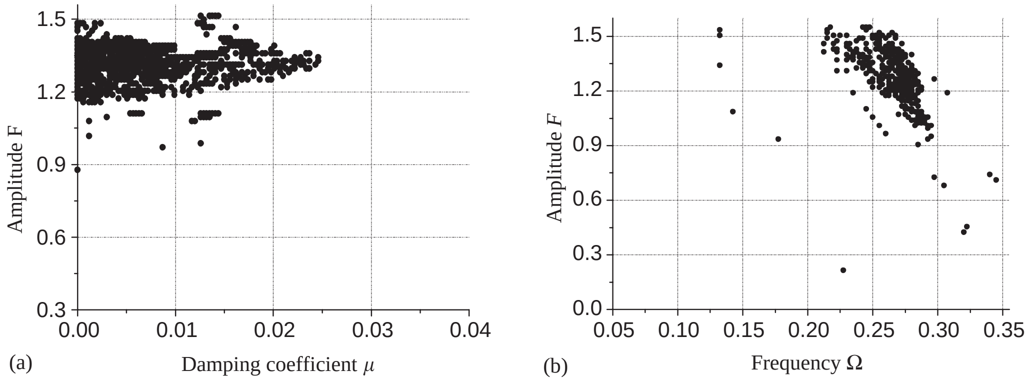

Fig. 2. Black regions and dots depict the domains where chaotic behaviour of the Bouc–Wen histeretic oscillator is possible ($k_z = 0.5$, $\gamma = 0.3$, $\beta = 0.005$, $n = 1.0$, $x(0) = 0.1$, $\dot{x}(0) = 0.1$, $z(0) = 0$): (a) in the (μ, F) plane ($\Omega = 0.27$); (b) in the (Ω, F) plane ($\mu = 0$). The hysteretic dissipation is characterized by the value $\delta = 0.0476$. In the case of motion without hysteretic dissipation only regular behaviour of the oscillator is possible.

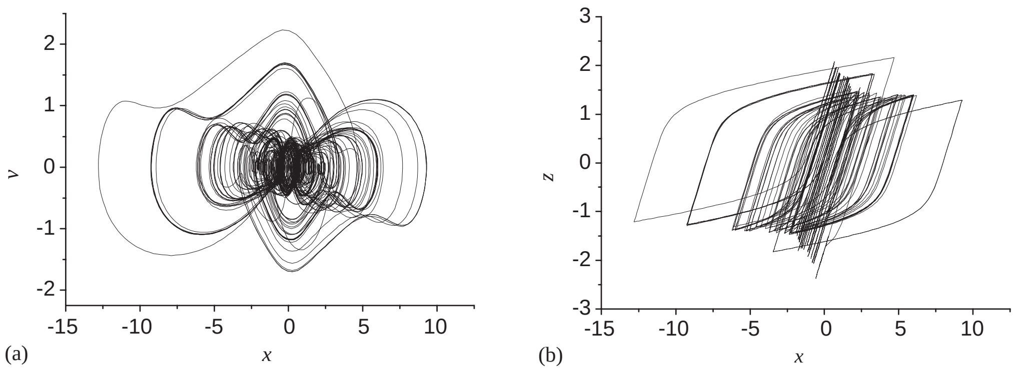

Fig. 3. Chaotic response of the Masing hysteretic oscillator ($\Omega = 0.15$, $F = 1.21$, $\mu = 0.026$, $v = 0.5$, $\delta = 0.05$, $n = 10.0$, $x(0) = 0.1$, $\dot{x}(0) = 0.1$, $z(0) = 0$): (a) phase portrait; (b) hysteresis loop.

Figs. 3 and 4 characterize the obtained domains and demonstrate oscillator motions of various character such as chaos and hysteresis loss (Fig. 3), and periodic response (Fig. 4).

3. Conclusions

Highly nonlinear hysteretic Masing and Bouc–Wen models with discontinuous right-hand sides are investigated using an effective approach based on analysis of the wandering trajectories [1–4]. This algorithm for quantifying regular and chaotic dynamics is simpler and faster from a computational point of view than standard procedures and is sufficiently accurate to trace regular/irregular responses of hysteretic systems. Domains where chaotic/regular behaviour of the oscillators with hysteresis is possible are found in planes determined by the damping

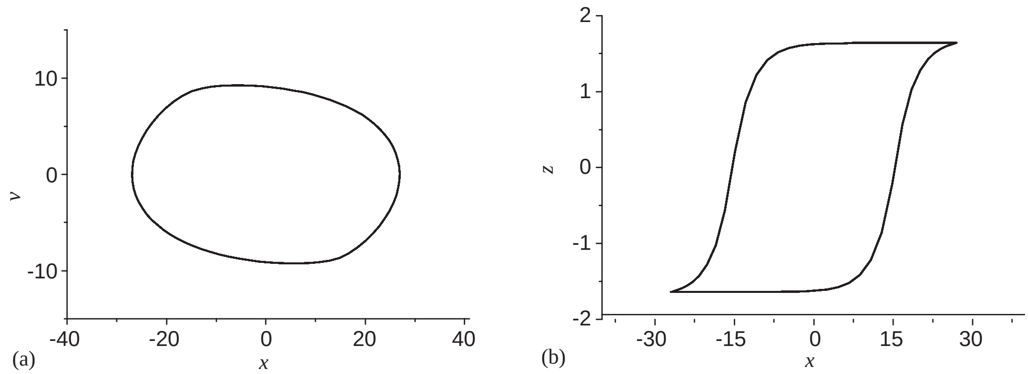

Fig. 4. Periodic response of the Bouc–Wen hysteretic oscillator ($\Omega = 0.35$; $F = 1.2$, $\mu = 0.0$, $\delta = 0.0476$, $k_z = 0.5$, $\gamma = 0.3$, $\beta = 0.005$, $n = 1.0$, $x(0) = 0.1$, $\dot{x}(0) = 0.1$, $z(0) = 0$): (a) phase portrait; (b) hysteresis loop.

coefficient—amplitude and frequency—amplitude regions under the external periodic excitation. A substantial influence of hysteretic dissipation on the possibility of chaotic behaviour occurrence in the systems with hysteresis is shown, and moreover, the restraining and generating effects of hysteretic dissipation on the occurrence of the chaotic behaviour is demonstrated.

Further investigations on the hysteretic dissipation generating effects on chaos occurrence are planned for the mechanical systems with memory modelled by means of additional state variables.

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